THE NIKOLAEVSKY EQUATION WITH DISPERSION

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ABSTRACT

The Nikolaevskiy equation was initially used to model longitudinal seismic waves in viscoelastic media. This equation shows spatiotemporal chaos at onset. In order to unfold the complicated dynamics dispersion is added and the effect on the solution is studied as the dispersion becomes small. Amplitude equations are derived and investigated in terms of the stability of travelling wave solutions (rolls) for different degrees of dispersion. For sufficiently large dispersion stable rolls exist. Regarding weak dispersion, a threshold point of the dispersion coefficient is calculated above which stable rolls exist for a fixed value of the bifurcation parameter. Finally, the stability problem is tested against numerical simulations.

Key Words: Nikolaevskiy equation, dispersion, Eckhaus instability, Ginzburg-Landau equation.

INTRODUCTION

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V. Nikolaevskiy (1989) introduced a model for seismic waves in viscoelastic media as follows:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \sum_{p=1}^{n} A_{p+1}\frac{\partial^{p+1} v}{\partial x^{p+1}},$$

where v is the velocity amplitude and A_{p+1} are constants. The standard rescaled form of this equation which is:

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2}\left[r - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right]u - u\frac{\partial u}{\partial x}, \tag{1}$$

is called the Nikolaevskiy equation. Moreover, r is the stability control parameter. The Nikolaevskiy equation is also used to describe other physical problems such as the phase equation for reaction-diffusion systems (Fujisaka & Yamada, 2001; Tanaka, 2004; Tanaka, 2006). In addition, it also describes transverse instabilities of fronts at finite wave number (Cox & Matthews, 2007) and condensed matter evaporated by a laser beam (Anisimov, Tribel'skiĩ, & Épel'baum, 1980). The turbulence in electroconvection of homeotropic nematic liquid crystals is analogous to the Nikolaevskiy equation (Anugraha, Hidaka, Oikawa, & Kai, 2008; Hidaka, Hun, Hayashi, Kai, & Tribelsky, 1997; Nagaya & Orihara, 2000; Tribel'skiĩ, 1997). The Nikolaevskiy equation is an interesting field of study due to its physical applications and chaotic solutions.

Numerical simulations of the Nikolaevskiy equation, for different values of the bifurcation parameter r, show sudden onset of complicated behaviour (see figure 1). This chaotic behaviour is similar to the Kuramoto-Sivashinsky equation (Kliakhandler & Malomed, 1997; Tanaka, 2005). Moreover, the spatiotemporal chaos at onset is due to the interaction between the short-wavelength modes in the unstable region and the long-wavelength modes near the wave number $k = 0$.

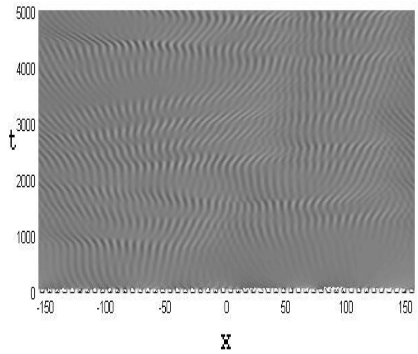


Figure 1: Space-time numerical simulations of the Nikolaevskiy equation with no dispersion ($r = 0.01$), where the real value of u is plotted.

Adding more terms to (1) helps to unfold the dynamics of the solution. This is because with extra terms new parameters are introduced and can be controlled. By this we can study the effect of adding more terms and how the solution is changed as the new parameters become small. Cox and Matthews (2007) added damping to (1) and studied the effect of damping on the solution. Stationary periodic solutions with wave number close to the critical value are stable for sufficiently strong damping. For a given value of r and sufficiently small damping, there is a critical value of damping above which stable stationary periodic solutions exist. If the damping is absent we get instability (Tribelsky & Velarde, 1996). Another damping term had been added to (1) by Tribelsky (2008), where the stability of steady spatially periodic patterns was studied for two degrees of damping. The damping terms presented by Cox and Matthews (2007) and Tribelsky (2008) are different, however numerical simulations exhibit similar results.

The original Nikolaevskiy model consisted of odd and even spatial derivatives. In most studies, only three terms of the even derivatives are kept. In some studies the two dispersion terms (third and fifth derivatives) are considered and analysed. The Nikolaevskiy equation with dispersion is as follows:

$$\frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2}\left[r - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right]u + \alpha\frac{\partial^3 \mathrm{u}}{\partial x^3} + \beta\frac{\partial^5 \mathrm{u}}{\partial x^5} - u\frac{\partial u}{\partial x}, \tag{2}$$

with periodic boundary conditions and α and β are the dispersion coefficients. The third order term was considered by Malomed (1992). In this study there was a derivation of the amplitude equations. Afterwards the amplitude equations were investigated in terms of the stability of travelling wave solutions (rolls). On the other hand Beresnev and Nikolaevskiy (1993) numerically simulated (2) and found that the fifth order term has no significant effect on the solution. Moreover, the third order term increases instability. On the contrary, Kudryashov and Migita (2007) found that there is a critical value of α (β) depending on r above which the simulations show stable periodic patterns.

All of the above analyses are incomplete regarding dispersion, so further investigation of adding both dispersion terms should be carried out.

Adding damping or dispersion to (1) helps to understand the dynamics of the solution as the degrees of damping (dispersion) can be varied. Two independent studies regarding damping were completed by Cox and Matthews (2007) and Tribelsky (2008). On the other hand, for dispersion there is not enough research done in this area. The aim of this study is to analyse the effect of adding dispersion to (1); by asymptotic expansions and numerical simulations. There will be an illustration of how different degrees of dispersion help to find stable rolls. The next section provides a brief description of the results found regarding dispersion.

THE NIKOLAEVSKY EQUATION WITH DISPERSION

The first step in analysing the solution of (2) is to linearise around the steady state solution $u \equiv 0$ and find solutions proportional to $e^{ikx+\lambda t}$. This implies the dispersion relation $\lambda = \lambda_r + i\lambda_i$, where $\lambda_r = k^2(r-(k^2-1)^2)$ and $\lambda_i = k^3(k^2\beta - \alpha)$. This dispersion relation shows that for $r > 0$ there exists a band of unstable wave numbers around $k = 1$ (see figure 2). Accordingly, the aim is to study the solution near $k = 1$ with $r = O(\varepsilon^2)$ $(\varepsilon \ll 1)$.

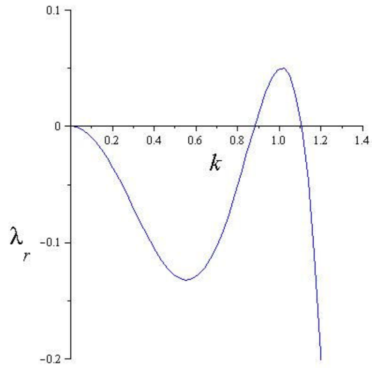
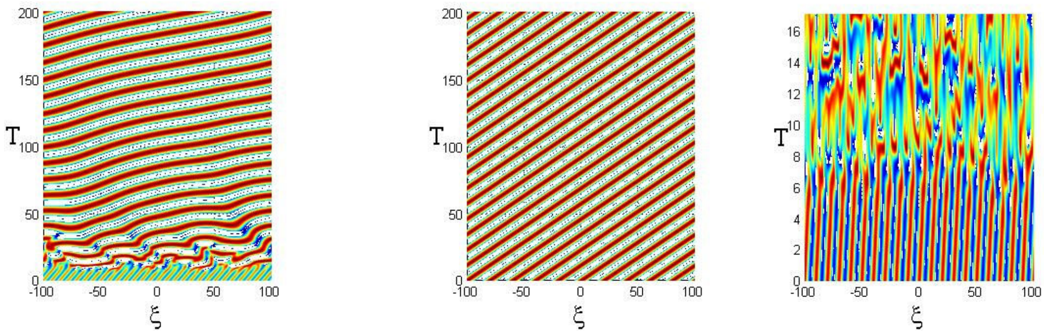


Figure 2: Plot of the growth rate λ_r for the case $r = 0.05$.

The coupled amplitude equations of the slowly varying envelopes are derived by weakly nonlinear analysis with α, β = *O* (1). These amplitude equations can be reduced to the complex Ginzburg-Landau equation. In addition, the stability analysis of the amplitude equation shows that dispersion can lead to an Eckhaus like band of stable wave numbers. Moreover, numerical simulations of the governing amplitude equation illustrate consistency with the stability analysis (see figure 3). However, for small values of the dispersion coefficients, another scaling is required. This is because the stability analysis suggests the existence of stable rolls for any values of α and β. However, it is known from the chaotic behaviour of the Nikolaevskiy equation with no dispersion that this analysis must break down for small values of dispersion. Therefore, different scaling is needed to resolve this apparent contradiction.



(a) α = 10, β = 2.6 and n = 28. (b) α = 10, β = 2.6 and n = 10. (c) α = 8.4, β = 2.6 and n = 20.

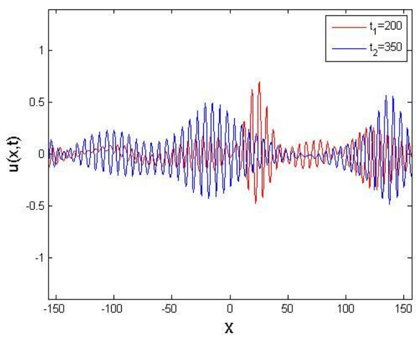
Figure 3: Space-time numerical simulations of the amplitude equation for strong dispersion, where the real value of the amplitude is plotted. The values of α, β and the number of waves n in the initial condition are given under each figure.

In the intermediate dispersion α and β are chosen to be $O(\varepsilon^{3/4})$ with a wave number close to 1, where the first scaling considered previously is the strong dispersion. After applying the weakly nonlinear regime on (2), the coupled amplitude equations are calculated. In addition, the stability analysis of these equations predicts that there are no stable travelling wave solutions. However, this analysis breaks down if the rolls have a wave number very close to 1 or close to the marginal stability curve. Therefore, these two cases need to be resolved separately (weak dispersion).

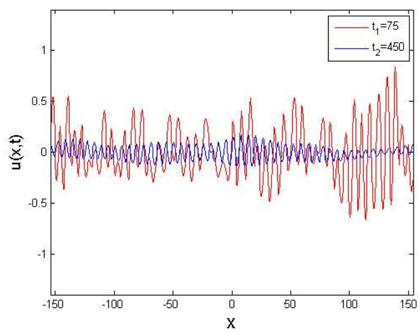
First the case of wave numbers very close to 1 is considered. This requires smaller values of α and β namely α, β = O (ε), again the coupled amplitude equations are found and the stability condition of the solution is calculated. The stability of travelling wave solutions regarding only one dispersion term is studied. The result is that there is a threshold point for α (β) such that any value of the dispersion parameter taken above this point will give a band of stable wave numbers.

In the other case regarding the region close to the marginal stability curve, the result is that there are no stable waves.

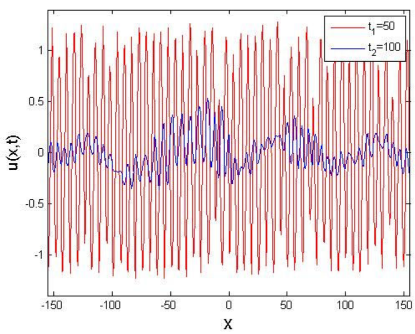
All of the previous analyses including the strong, intermediate and weak dispersion are tested against numerical simulations of (2). In these simulations certain values of α and β are chosen according to different degrees of dispersion. Also r is chosen with rolls having a wave number either from the stable or unstable region predicted by the asymptotic results. Figure 4 illustrates some numerical simulations of (2). These solutions show good agreement with the theoretical results.



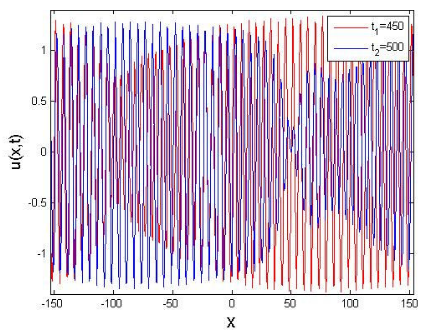
(a) α = 2, β = 1, r = 0.01 and k = 1.



(b) α = 0.3557, β = 0.1778, r = 0.01 and k = 1.02.



(c) α = 0.2, β = 0, r = 0.01 and k = 1.0087.



(d) α = 0, β = 0.5, r = 0.01 and k = 1.025.

Figure 4: Snap shots of the numerical simulations of (2) with dispersion values and wave number (k) as indicated under each graph. The snap shots are taken in two different times as given above each figure.

CONCLUSION

The Nikolaevskiy equation is widely studied because of its physical applications and complicated solution. Numerous studies were carried out in order to unfold the chaotic behaviour. These studies consisted of numerical simulations of the Nikolaevskiy equation for $r > 0$. In addition, amplitude equations were achieved by asymptotic expansions near the critical wave number for small r and analysed in terms of the stability of rolls.

In this study, dispersion terms were added to the Nikolaevskiy equation and investigated for different rates of dispersion. The complex Ginzburg-Landau equation resulted from the strong dispersion which leads to the existence of an Eckhaus like band of stable wave numbers. This analysis broke down for small values of α and β and therefore another scaling was required for the intermediate dispersion. This resulted in no stable rolls. Also this analysis was not valid for wave numbers very close to the critical value or near the marginal stability curve. Close to the critical wave number, a threshold value of α and β were found separately such that stable waves exist if the dispersion coefficient is taken above this critical number. On the other hand, if the rolls have a wave number close to the marginal stability curve, then they are unstable. All of the above analyses were tested against numerically computed solutions and showed agreement between them.

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